On the Monotonicity of Work Function in k-Server Conjecture

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Abstract

This paper presents a mistake in work fuction algorithm of k-server conjecture. That is, the monotonicity of the work fuction is not always true.

1 Introduction

The k-server conjecture has not been proved. A lot of literature deal with the k-server conjecture [3] [2] [4] [1] and references therein. In [3], the work function algorithm (WFA) is so far the best determined algorithm for this problem. In [3], there are the facts as follows (excerpts from [3]):

Fact 3 For a work function w and two configurations X, Y

$$w(X) \leq w(Y) + D(X, Y) \tag{1}$$

Consider a work function w and the resulting work function w' after request r. By Fact 3 we get

$$w'(X) = \min_{x \in X}\{w(X - x + r) + d(r, x)\} \geq w(X) \tag{2}$$

which translats to:
Fact 4 Let w be a work function and let w' be the resulting work function after request r. Then for all configurations X :

$$w'(X) \geq w(X) \tag{3}$$

But Fact 4 is not true. That is, the monotonicity of work function is not true.

2 The monotonicity of work function

Fact 4 is not true because (2) is incorrect. From

$$w'(X) = \min_{x \in X}\{w'(X - x + r) + d(r, x)\} \tag{4}$$

we can get (it is Fact 3, here it is true):

$$w'(X) \leq w'(X - x + r) + d(r, x) \tag{5}$$

That is (because $w'(X - x + r) = w(X - x + r)$):

$$w'(X) \leq w(X - x + r) + d(r, x) \tag{6}$$

Assume that Y is a configuration which makes the minimum of $w'(X)$, so

$$w'(X) = w(Y) + D(X, Y) \tag{7}$$

But we cannot get $w'(X) \geq w(X)$ (Fact 4) from (7) based on Fact 3 because Fact 3 is incorrect for this case. If Fact 3 is derived from $w(X) = \min_{x \in X}\{w(X - x + r) + d(r, x)\}$, it is true. But it cannot be used universally in all other cases because Fact 3 is derived under some conditions
It is known

$$w(X) = \min_{x \in X}\{w(X - x + r') + d(r', x)\} \tag{8}$$

where r' is the request before the request r. Assume that Z is a configuration which makes the minimum of $w(X)$, so

$$w(X) = w(Z) + D(X, Z) \tag{9}$$

In order for Fact 4 to be true, we have to prove the following:

$$w'(X) = w(Y) + D(X, Y) \geq w(Z) + D(X, Z) \tag{10}$$

Unfortunately, the above is not always true.
We give a concrete counterexample as follows.
A 5-node weighted undirected graph. The node set is a, b, c, d, e. The distances (edge weights) are as follows.

$$d(a, b) = 1, d(a, c) = 7, d(a, d) = 5, d(a, e) = 8, d(b, c) = 4,$$

$$d(b, d) = 2, d(b, e) = 10, d(c, d) = 3, d(c, e) = 9, d(d, e) = 6$$

Consider 3-servers on this graph. The initial configuration is abc and the request sequence are

$$e, d, a, c, b, d$$

In the folloing table we give values of work functions corresponding to all 3-node configurations and all request sequence.

Table: Values of Work Functions for 3-servers

		Configuration									
Request	i	*abc*	*abd*	*abe*	*acd*	*ace*	*ade*	*bcd*	*bce*	*bde*	*cde*
ϕ	0	0	3	9	2	10	11	5	8	11	10
e	1	16	15	9	16	10	11	14	8	11	10
d	2	18	15	13	16	14	11	14	12	11	10
a	3	18	15	13	16	14	11	17	15	12	18
c	4	18	20	18	16	14	17	17	15	18	18
b	5	18	20	18	18	16	19	17	15	18	17
d	6	20	20	21	18	22	19	17	19	18	17

From the above table we can see

$$w_{edacb}(cde) < w_{edac}(cde)$$

so the Fact 4 is not always true. That is, the work function does not have the monotonicity. In paper [3], all theorems which are proved based on the monotonicity have to be re-examined. In paper [3], the extended cost may overestimate the online cost. We still think WFA would be k-competitive.

References

[1] Allan Borodim and Ran El-Yaniv. *Online Computation and Competitive Analysis.* Cambridge University Press, 1998.

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[4] Lawrence L. Larmore and Lames A. Oravec. T-theory applications to online algorithms for the server problem. *arXiv:cs/0611088v1 [cs.DS] 18 Nov*, 2006.